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The Highest & Lowest Reliability Achievable with Redundancy

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Key Words—Redundancy, s -Independence, Theta transformation.

Reader Aids—

Purpose: Advance the state of the art

Special math needed for explanations & results: Probability

Results useful to: Reliability engineers and theoreticians

Summary & Conclusions—Often system reliability can be enhanced through the use of redundancy. Redundancy may, however, have a detrimental effect on the statistical relationship of redundant elements. When the components in a redundant system have more than one failure-mode and when failures are s -dependent, it is difficult to assess the reliability of the system. The paper describes the θ -transformation by which the highest and lowest reliability achievable can be determined for a configuration using components with specified reliabilities. As a by-product we become able to pinpoint the statistical relationships that give rise to highest and lowest system reliabilities.

1. INTRODUCTION

In many engineering applications, the designer must have some multivariate probability distribution specified before he can solve the problem at hand, but he has access only to the N -marginal distributions rather than to the complete N -variate distribution. The reason for this is that practically all available information is univariate, e.g. tables from statistical yearbooks, and vendor information on the true parameter values of their components. In such cases it therefore becomes important for the designer to determine the most favorable and the least favorable N -variate distribution concomitant with N -specified marginals. Given the extreme Cdf's, he can then find the maximum and minimum value for the variable of interest. To solve his problem, the designer must be familiar with the θ -transformation whereby all N -variate distributions that are concomitant with a set of N -specified marginals can be determined. The θ -transformation was developed by the author [1]; it presupposes only that all N -marginal Cdf's are discrete and that all probabilities are rational numbers. Once familiar with the θ -transformation, the designer can search for a most (or least) favorable N -variate Cdf using hill-climbing techniques. The θ -transformation is a tool of wide applicability; e.g. in classification problems it can be used to pinpoint the N -variate Cdf's that result in the largest (or smallest) classification error, given the marginals [2]. In this paper the θ -transformation is used to compute the highest and lowest

reliability achievable with some redundant configurations. Further applications of the θ -transformation can be found in [3]. The paper is organized as follows. In Section 2 the θ -transformation is described in some detail. In Section 3 and 4, by way of illustrative examples, we study the reliability enhancement (and degradation) achievable with double and triple active redundancy applied to diodes.

2. MULTIVARIATE DISTRIBUTIONS WITH SPECIFIED MARGINALS

2.1 Introduction

This section answers the following question: given a multivariate Cdf and all its marginal Cdf's how does one find the other multivariate Cdf's that have the same set of marginals? The answer is: all multivariate Cdf's which have a specified set of marginals, are obtained by repeated application of the θ -transformation to the product of the marginals. The practical value of the answer is that it now becomes possible to determine what the 'best' and the 'worst' multivariate Cdf's (in some particular sense) are for a specified set of marginals. With knowledge of these extreme multivariate Cdf's the designer can bound variables of interest.

2.2 Preliminary Assumptions

1. An experiment has been repeated a large number of times. At the end of each repetition we measure the values of n parameters;

$$\mathbf{x} \equiv (x_1, \dots, x_n).$$

2. $x_j, j = 1, \dots, n$, always take one of n_j discrete, fixed values. The assumption is satisfied in practice due to the inherent quantization and limited range of results obtained with measuring equipment. Consequently, the experiment (as judged from the parameter values) can have no more than

$$n_p = n_1 \cdot \dots \cdot n_n \quad (1)$$

different outcomes, each of which is a lattice point in the n -dimensional measurement space. With each lattice point is associated the relative frequency of the corresponding result of the experiment.

3. The multivariate pmf for \mathbf{x} , $f \equiv f(\mathbf{x})$, can be estimated from these relative frequencies, an assumption which is widely used in engineering.

4. Each of the n_p discrete probabilities that constitute f is a multiple of some small quantity of probability mass, q (this is true for all sets of n_p rational numbers). The assumption is justified by the finite accuracy of the measuring and computing devices.

f has n marginal pmf's. They are called $f_j = f_j(x_j)$, $j = 1, \dots, n$; f_j consists of n_j discrete probabilities having a sum of unity. The product of the n marginals is f_p ; f_p is a multivariate pmf, and like f it has the n marginals f_j , $j = 1, \dots, n$. f_p is the multivariate pmf that results when x_1, \dots, x_n are s -independent variables.

2.3 The θ -Transformation

Lattice points are designated by P_i . Consider two lattice points: $P_1 \equiv (a_1, \dots, a_n)$, and $P_2 \equiv (a_1 + b_1, \dots, a_n + b_n)$; the two values of \mathbf{x} are observed with probabilities θ_1 and θ_2 . $0 < \theta < \min \{\theta_1, \theta_2\}$. The θ -transformation consists of moving θ units of probability mass from P_1 to P_3 , while simultaneously moving θ units of probability mass from P_2 to P_4 . The locations of P_3 and P_4 are determined as follows. The n coordinates for P_3 are obtained by using some of the P_1 coordinate-values and some of the P_2 coordinate-values; x_j is a_j or $(a_j + b_j)$. The above-mentioned coordinate-values not used for P_3 are used for P_4 . For illustration, consider the case $n = 3$; P_3 could be $(a_1, a_2, a_3 + b_3)$, then P_4 would be $(a_1 + b_1, a_2 + b_2, a_3)$. The θ -transformation leaves the n marginals unchanged because the probability of a lattice-point having $x_j = a_j$, $f_j(a_j)$, or having $x_j = a_j + b_j$, $f_j(a_j + b_j)$, $j = 1, \dots, n$, is unchanged.

The reader can verify that the above statement is true for $n = 2$.

2.4 A Theorem

We are concerned with joint pmf's that consist of n_p discrete probabilities; such joint pmf's will be called 'admissible'. The set of n marginals is 'admissible' because all marginals consist of discrete probabilities; f_p , the product of the n members of an admissible set of marginals, clearly is an admissible joint pmf. An admissible joint pmf always has an admissible set of n marginals. An 'admissible sequence of joint pmf's' is a sequence of admissible joint pmf's f^1, f^2, f^3, \dots each of which is obtained from its predecessor by exactly one θ -transformation. When n_p and q are specified, the number of possible, admissible joint pmf's obviously is bounded. The theorem can now be stated; for a proof see [1].

Theorem: Let f be a discrete admissible joint pmf with the marginals (f_1, \dots, f_n) . There exists at least one finite-length admissible sequence of joint pmf's that begins with f_p and ends with f .

The importance of the theorem is that it insures against the following unpleasant possibility. "In order to reach an optimum admissible joint pmf by hill-climbing from f_p , it is necessary to pass through nonadmissible joint pmf's." The theorem says that this can never happen.

The theorem does not tell the hill-climbing designer how to find the multivariate pmf that has some particular property in largest measure; the theorem only states that the interesting multivariate pmf can be generated from f_p by a finite number of applications of the θ -transformation, each of which changes one admissible pmf to another admissible pmf.

2.5 Continuous Distributions

If f is continuous, rather than discrete as hitherto assumed, the situation changes somewhat. As before, we have the n marginals and their product f_p , all of which are continuous. The θ -transformation now takes the following form. We select four points, P_1, P_2, P_3 and P_4 , as described in the beginning of section 2.3. At each point, and in the same manner, we locate identical orthotopes (n -dimensional boxes with sides vertical to the axis) O_1, O_2, O_3, O_4 . We redefine θ as some continuous function defined over the orthotope. If we now substitute f^k with $(f^k - \theta)$ inside O_1 and O_2 , and with $(f^k + \theta)$ inside O_3 and O_4 , it is seen that the marginals are retained. If $(f^k - \theta)$ inside O_1 and O_2 , and $(f^k + \theta)$ inside O_3 and O_4 , all are non-negative functions, the θ -transformation is admissible. The main difference between the discrete and the continuous case is that in the continuous case there is no assurance that a finite number of θ -transformations can change f_p to f as stated in the Theorem.

3. EXAMPLE 1: TWO DIODE CIRCUITS WITH ACTIVE REDUNDANCY

A diode is located in a hostile environment for some specified length of time. We want to improve the reliability through using redundancy. We know from experiments that the diode states and probabilities are as follows.

TABLE A
Diode States

| | state | prob. |
|---------|-------|-------------|
| correct | c | $p_c = 0.7$ |
| open | o | $p_o = 0.1$ |
| short | s | $p_s = 0.2$ |

The 2 failure modes are catastrophic; the 3 states are mutually exclusive and exhaustive.

Parallel Redundancy. Consider using 2 diodes, D_1 & D_2 , in parallel. If one diode opens, the other diode makes the parallel combination still work correctly (see Table 1). On the other hand, if either diode shorts, the parallel combination shorts (see Table 1). Do we gain anything by using parallel redundancy? Table 1 shows the 9 possible states for the diode pair, the probabilities of those states, and totals for the pair-states. If the states of the diodes are s -independent, the 9

joint pmf's are the 9 numbers in the 'usual' column; the parallel combination will be $49\% + 7\% + 7\% = 63\%$ correct which is less than $p_c = 0.7$ for a single diode.

Now let us apply the θ -transformation three times. First we change the 2 probabilities 0.02 to 0.00 while at the same time changing 0.04 and 0.01 to 0.06 and 0.03 (a check shows that the 2 marginals stay the same just as they should). Second, we change the 0.14's to 0.00's while the same time changing 0.49 and 0.06 to 0.63 and 0.20. Finally, we change the 0.07's to 0.10's while at the same time changing 0.63 and 0.03 to 0.60 and 0.00. Using the θ -transformation (thrice), we have arrived at the 9 joint pmf's which are listed in the 'best' column; the pair reliability has increased to $60\% + 10\% + 10\% = 80\%$. This is the highest reliability possible, because no θ -transformation can increase the figure above 80%. An inspection shows that this maximum reliability is achieved when the two diodes states are s -dependent in such a manner that both diodes always short-circuit together but never open-circuit together.

If on the other hand, the s -dependence is such that both diodes always open-circuit at the same time but never short-circuit at the same time, repeated use of the θ -transformation yields the joint pmf's in the 'worst' column. No θ -transformation can bring the reliability below $50\% + 0\% + 0\% = 50\%$.

In summary, the reliability of the two diodes in parallel can be anywhere from 50% to 80%, depending on the statistical relationships between the states of the two diodes.

Series Redundancy. If we connect the two diodes D_1 and D_2 in series, we can go through a similar set of arguments. Table 2 illustrates the situation. The reliability of the series connection can be anywhere between 90% and 60% depending on the statistical relationship between the states of the two diodes. The number in the 'best' column in Table 1 correspond to those in the 'worst' column in Table 2 *et vice versa*. The reason for this is that the least (most) desirable statistical relationship between the states of D_1 and D_2 in the parallel configuration is the most (least) desirable in the series configuration.

TABLE 1
2 Parallel Diodes
($p_c = 0.7, p_s = 0.2, p_o = 0.1$)

| states | | | joint pmf | | |
|--------|-------|------|-----------|------|-------|
| D_1 | D_2 | pair | usual | best | worst |
| c | c | c | .49 | .60 | .50 |
| c | o | c | .07 | .10 | .00 |
| c | s | s | .14 | .00 | .20 |
| o | c | c | .07 | .10 | .00 |
| o | o | o | .01 | .00 | .01 |
| o | s | s | .02 | .00 | .00 |
| s | c | s | .14 | .00 | .20 |
| s | o | s | .02 | .00 | .00 |
| s | s | s | .04 | .20 | .00 |
| totals | | | | | |
| c | | | .63 | .80 | .50 |
| o | | | .01 | .00 | .10 |
| s | | | .36 | .20 | .40 |

TABLE 2
2 Series Diodes
($p_c = 0.7, p_s = 0.2, p_o = 0.1$)

| states | | | joint pmf | | |
|--------|-------|------|-----------|------|-------|
| D_1 | D_2 | pair | usual | best | worst |
| c | c | c | .49 | .50 | .60 |
| c | o | o | .07 | .00 | .10 |
| c | s | c | .14 | .20 | .00 |
| o | c | o | .07 | .00 | .10 |
| o | o | o | .01 | .10 | .00 |
| o | s | o | .02 | .00 | .00 |
| s | c | c | .14 | .20 | .00 |
| s | o | o | .02 | .00 | .00 |
| s | s | s | .04 | .00 | .20 |
| totals | | | | | |
| c | | | .77 | .90 | .60 |
| o | | | .19 | .10 | .20 |
| s | | | .04 | .00 | .20 |

Less Pessimistic or Optimistic Worst-Cases. In the examples illustrated in Tables 1 and 2 we have shown how the θ -transformation can be used to pinpoint the 'most favorable' and 'least favorable' bivariate pmf's and thereby establish the ranges of reliability achievable by parallel and series redundancy. At this point, the reader may rightfully object that the worst-case situations are unnecessarily pessimistic. In Table 1 it is, for instance, not realistic to assume that the joint probability of one diode's being open while the other diode functions correctly should be zero, i.e. that the very situation that justifies using parallel redundancy is impossible.

When, therefore, θ -transformations are used to determine worst-case system-reliabilities the designer should abstain from transformations which result in joint events having unrealistic joint probabilities. This may be readily achieved by putting upper and lower bounds on the joint probabilities.

4. EXAMPLE 2: A THIRD CIRCUIT WITH REDUNDANCY

Table 3 illustrates three diodes in parallel, an elaboration of the 2-diode configuration from Table 1. In this section we will use the θ -transformation to determine the smallest achievable reliability with this configuration; at the same time we will determine the statistical relationships that make this minimum reliability a reality.

As before, we assume that each diode, D_1, D_2, D_3 , has the properties in Table A *et seq.* The 3-diode triplet will be in one of $3^3 = 27$ possible states which are mutually exclusive and exhaustive. Table 3 illustrates the 27 triplet states and the state of the triplet. It also shows the associated probabilities as in Tables 1 & 2. If the diode states are s -independent, the 27 states of the triplet will have the probabilities listed in the 'usual' column of Table 3.

If we remove 0.028 units of probability mass from (c, c, c) and from (c, s, s) while adding 0.028 units at (c, c, s) and (c, s, c) , the marginals remain the same for D_1, D_2 and D_3 . The θ -transformation we will describe as:

TABLE 3
3 Parallel Diodes
($p_c = 0.7, p_s = .0.2, p_o = 0.1$)

| states | | | | joint pmf | |
|--------|-------|-------|---------|-----------|-------|
| D_1 | D_2 | D_3 | triplet | usual | worst |
| c | c | c | c | .343 | .315 |
| c | c | o | c | .049 | .042 |
| c | c | s | s | .098 | .129 |
| c | o | c | c | .049 | .042 |
| c | o | o | c | .007 | .007 |
| c | o | s | s | .014 | .018 |
| c | s | c | s | .098 | .129 |
| c | s | o | s | .014 | .018 |
| c | s | s | s | .028 | .000 |
| o | c | c | c | .049 | .000 |
| o | c | o | c | .007 | .000 |
| o | c | s | s | .014 | .045 |
| o | o | c | c | .007 | .000 |
| o | o | o | o | .001 | .010 |
| o | o | s | s | .002 | .000 |
| o | s | c | s | .014 | .045 |
| o | s | o | s | .002 | .000 |
| o | s | s | s | .004 | .000 |
| s | c | c | s | .098 | .148 |
| s | c | o | s | .014 | .021 |
| s | c | s | s | .028 | .000 |
| s | o | c | s | .014 | .021 |
| s | o | o | s | .002 | .002 |
| s | o | s | s | .004 | .000 |
| s | s | c | s | .028 | .000 |
| s | s | o | s | .004 | .000 |
| s | s | s | s | .008 | .008 |
| totals | | | | | |
| c | | | | .511 | .406 |
| o | | | | .001 | .010 |
| s | | | | .488 | .584 |

TABLE 4

Let f_p indicate the 27 discrete probabilities corresponding to the states of D_1, D_2 , and D_3 being s -independent. By applying the 10 θ -transportations listed above to f_p we arrive at the trivariate pmf which minimizes the reliability of the 3-diode configuration in Table 3.

| | | | | | | | | | | | | | |
|------------|------------|------------|-----------|---|---------|--------|--------|--------|-----|---------|--------|--------|--------|
| (c, c, c), | (c, s, s); | (c, c, s), | (c, s, c) | : | (0.343, | 0.028; | 0.098, | 0.098) | = > | (0.315, | 0.000; | 0.126, | 0.126) |
| (o, o, s), | (o, s, o); | (o, s, s), | (o, o, o) | : | (0.002, | 0.002; | 0.004, | 0.001) | = > | (0.000, | 0.000; | 0.006, | 0.003) |
| (o, o, c), | (o, c, o); | (o, c, c), | (o, o, o) | : | (0.007, | 0.007; | 0.049, | 0.003) | = > | (0.000, | 0.000; | 0.056, | 0.010) |
| (o, c, c), | (o, s, s); | (o, s, c), | (o, c, s) | : | (0.056, | 0.006; | 0.014, | 0.014) | = > | (0.050, | 0.000; | 0.020, | 0.020) |
| (o, c, c), | (s, s, c); | (o, s, c), | (s, c, c) | : | (0.050, | 0.028; | 0.020, | 0.098) | = > | (0.025, | 0.003; | 0.045, | 0.123) |
| (o, c, c), | (s, c, s); | (o, c, s), | (s, c, c) | : | (0.025, | 0.028; | 0.020, | 0.123) | = > | (0.000, | 0.003; | 0.045, | 0.148) |
| (c, c, o), | (s, s, o); | (c, s, o), | (s, c, o) | : | (0.049, | 0.004; | 0.014, | 0.014) | = > | (0.045, | 0.000; | 0.018, | 0.018) |
| (c, o, c), | (s, o, s); | (s, o, c), | (c, o, s) | : | (0.049, | 0.004; | 0.014, | 0.014) | = > | (0.045, | 0.000; | 0.018, | 0.018) |
| (c, o, c), | (s, s, c); | (s, o, c), | (c, s, c) | : | (0.045, | 0.003; | 0.018, | 0.126) | = > | (0.042, | 0.000; | 0.021, | 0.129) |
| (c, c, o), | (s, c, s); | (s, c, o), | (c, c, s) | : | (0.045, | 0.003; | 0.018, | 0.126) | = > | (0.042, | 0.000; | 0.021, | 0.129) |

$[(c, c, c), (c, s, s); (c, c, s), (c, s, c)] :$

$(0.343, 0.028; 0.098, 0.098) \rightarrow$

$(0.315, 0.000; 0.126, 0.126)$

The θ -transformation results in new probabilities illustrating changed statistical relationships among D_1, D_2, D_3 . With the new probabilities — specifically the new probability of the event (c, c, c) — the reliability of the configuration has decreased by 0.028. We will now search for the lowest reliability possible with the configuration by applying a number of θ -transformations. The θ -transformations are listed in Table 4. A set of 27 probabilities that result in minimum reliability of the configuration are listed in the 'worst' column of Table 3. The minimum triplet reliability is 0.406 $(0.315 + 0.042 + 0.042 + 0.007 + 0 + 0 + 0)$. Other sets can be obtained, e.g. by adding θ units of probability mass at (c, c, c) and (c, o, o) and subtracting θ at (c, o, c) and (c, c, o) .

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BIOGRAPHY

Dr. Peter W. Becker (A'62) received his MS and dr. techn. degree in Electrical Engineering from the Technical University of Denmark. He is a member of the staff at the Electronics Laboratory of the Technical University of Denmark. His previous technical experience includes an 8-year period with the General Electric Co., Electronics Laboratory at Syracuse. He is interested in pattern recognition and published (1968) *Recognition of Patterns Using the Frequency of Occurrence of Binary Words*; a second, revised edition was published by Springer-Verlag 1974. Later he published *An Introduction to the Design of Pattern Recognition Devices*, Springer-Verlag 1972. Dr. Becker has written several papers in the field of reliability. In 1974 he published *Design of Systems and Circuits for Maximum Reliability or Maximum Production Yield* (co-authored by Mr. Finn Jensen); a revised edition is being published by McGraw-Hill, a Russian translation will be published soon. He serves on the editorial board of the International Journal of General Systems.

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